

# On Backus average for oblique incidence

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## Abstract

We postulate that validity of the Backus (1962) average, whose weights are layer thicknesses, is limited to waves whose incidence is nearly vertical. The accuracy of this average decreases with the increase of the source-receiver offset. However, if the weighting is adjusted by the distance travelled by a signal in each layer, such a modified average results in accurate predictions of traveltimes through these layers.

## 1 Introduction

Hookean solids, which are commonly used in seismology as mathematical analogies of physical materials, are defined by their mechanical property relating linearly the stress tensor,  $\sigma$ , and the strain tensor,  $\varepsilon$ ,

$$\sigma_{ij} = \sum_{k=1}^3 \sum_{\ell=1}^3 c_{ijkl} \varepsilon_{k\ell}, \quad i, j = 1, 2, 3,$$

where  $c$  is the elasticity tensor. The Backus (1962) average allows us to quantify the response of a wave propagating through a series of parallel Hookean layers whose thicknesses are much smaller than the wavelength.

According to Backus (1962), the average of  $f(x_3)$  of “width”  $\ell'$  is

$$\bar{f}(x_3) := \int_{-\infty}^{\infty} w(\zeta - x_3) f(\zeta) d\zeta, \quad (1)$$

where  $w(x_3)$  is the weight function with the following properties:

$$w(x_3) \geq 0, \quad w(\pm\infty) = 0, \quad \int_{-\infty}^{\infty} w(x_3) dx_3 = 1, \quad \int_{-\infty}^{\infty} x_3 w(x_3) dx_3 = 0, \quad \int_{-\infty}^{\infty} x_3^2 w(x_3) dx_3 = (\ell')^2.$$

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These properties define  $w(x_3)$  as a probability-density function with mean 0 and standard deviation  $\ell'$ , explaining the use of the term “width” for  $\ell'$ .

The long-wavelength homogeneous media equivalent to a stack of isotropic or transversely isotropic layers with thicknesses much less than the signal wavelength are shown by Backus (1962) to be transversely isotropic. The Backus (1962) formulation is reviewed by Slawinski (2016) and Bos et al. (2016), where formulations for generally anisotropic, monoclinic, and orthotropic thin layers are also derived. Bos et al. (2016) examine assumptions and approximations underlying the Backus (1962) formulation, which is derived by expressing rapidly varying stresses and strains in terms of products of algebraic combinations of rapidly varying elasticity parameters with slowly varying stresses and strains. The only mathematical approximation in the formulation is that the average of a product of a rapidly varying function and a slowly varying function is approximately equal to the product of the averages of the two functions.

## 2 Formulation

Let us consider a stack of ten isotropic horizontal layers, each with a thickness of 100 meters (Brisco, 2014). Their elasticity parameters are listed in Appendix A, Table 1.

For vertical incidence, the Fermat traveltimes through these layers is 229.47 ms. If we perform the standard Backus average—weighted by layer thickness of these ten layers, as in equation (2), below—then, the equivalent density-scaled elasticity parameters are  $\langle c_{1111} \rangle = 18.84$ ,  $\langle c_{1212} \rangle = 3.99$ ,  $\langle c_{1133} \rangle = 10.96$ ,  $\langle c_{2323} \rangle = 3.38$  and  $\langle c_{3333} \rangle = 18.43$ ; their units are  $10^6 \text{m}^{-2}\text{s}^{-2}$ . With these parameters and for the vertical incidence, the resulting  $P$ -wave traveltimes through the equivalent transversely isotropic medium is 232.92 ms, which—in comparison to the Fermat traveltimes—is high by 3.45 ms.

To examine the layer-thickness weighting, let us consider one of the equivalent-medium parameters,

$$c_{1212}^{\overline{\text{TI}}} = \frac{\sum_{i=1}^n h_i c_{2323i}}{\sum_{j=1}^n h_j}, \quad (2)$$

where  $h_i$  is the thickness of the  $i$ th layer, which herein is 100 m for each layer; thus, each layer is weighted equally by 0.1.

If we consider a  $P$ -wave signal whose takeoff angle, with respect to the vertical, is  $\pi/6$ , this signal reaches—in accordance with Snell’s law—the bottom of the stack at a horizontal distance of 1072.89 m. Its Fermat traveltimes is 330.58 ms.

If we perform the standard Backus average, the traveltimes in the equivalent medium, which corresponds to the ray angle of  $47.01^\circ$ , is 343.87 ms, which is higher by 13.3 ms than its Fermat counterpart. If, however, we weight the average by the distance travelled in each layer, as in equation (3), below, the equivalent elasticity parameters become  $\langle c_{1111} \rangle = 20.126$ ,  $\langle c_{1212} \rangle = 4.100$ ,  $\langle c_{1133} \rangle = 12.059$ ,  $\langle c_{2323} \rangle = 3.450$  and  $\langle c_{3333} \rangle = 19.762$ . In such a case, the traveltimes is 332.44 ms—which is higher by only 1.9 ms—and is an order of magnitude more accurate than

using the standard approach. The distances travelled in each layer and the resulting weights are given in Appendix A, Table 2. In such a case, expression (2) becomes

$$\overline{c_{1212}^{\text{TI}}} = \frac{\sum_{i=1}^n d_i c_{2323i}}{\sum_{j=1}^n d_j}, \quad (3)$$

where  $d_i$  is the distance travelled in the  $i$ th layer, which—for vertical incidence—is equal to  $h_i$ .

According to Lemma 2 of Bos et al. (2016), the stability conditions are preserved by the Backus average. In other words, if the individual layers satisfy these conditions, so does their equivalent medium. This remains true for the modified Backus average.

### 3 Discussion

The Backus (1962) average with weighting by the thickness of layer assumes vertical or near-vertical incidence. Consequently, such an average does not result in accurate traveltimes for the far-offset or, in particular, cross-well data, which nowadays are common seismic experiments, and were not half-a-century ago, when the Backus (1962) average was formulated.

If we modify the weighting to be by the distance travelled in each layer, then the resulting traveltimes are significantly more accurate. Such weighting, however, entails further considerations. Since the distance travelled in each layer is a function of Snell’s law, there is a need to modify the weights with the source-receiver offset. However, given information about layers, it is achievable algorithmically by accounting for distance travelled in each layer as a function of offset.

There is also an interesting issue to consider. The modified equivalent medium is defined by its elasticity parameters, which are functions of the obliqueness of rays within each layer. This means that the equivalent-medium parameters are different for the  $qP$  waves, for the  $qSV$  waves and for the  $SH$  waves. However, since a Hookean solid exists in the mathematical realm, not the physical world, such a consideration is not paradoxical. It is common to invoke even different constitutive equations for the same physical material depending on empirical considerations. Furthermore, it might be possible to derive elasticity parameters of a single Hookean solid—possibly of a material symmetry lower than transverse isotropy—whose behaviour accounts for both near and far offsets in the case of three waves.

It is interesting to note that—in each examined case—the traveltime in the equivalent medium is greater than its Fermat counterpart through the sequence of layers. It might be a consequence of optimization, which—in the case of layers—benefits from a model with a larger number of parameters.

There remains a fundamental question: Is the Fermat traveltime an appropriate criterion to consider the accuracy of the Backus average? An objection to such a criterion is provided by the following *Gedankenexperiment*. Consider a stack of thin layers, where—in one of these layers—waves propagate much faster than in all others. In accordance with Fermat’s principle, distance travelled by a signal within this layer is much larger than in any other layer, which might be expressed by the ratio of a distance travelled in a given layer divided by its thickness. This effect

is not accommodated by the standard Backus average, since this effect is offset-dependent and the average is not, but it is accommodated by the modified average discussed herein. However, a property of such a single layer might be negligible on long-wavelength signal. To address such issues, it might be necessary to consider a full-waveform forward model, and even a laboratory experimental set-up.

As an aside, let us recognize that—if we keep the Fermat traveltime as a criterion—making the propagation speed a function of the wavelength would not accommodate the traveltime discrepancy due to offset.

Be that as it may, it must be recognized that the discrepancy between the traveltimes in the layered and equivalent media increases with the source-receiver offset. In the limit—for a wave propagating horizontally through a stack of horizontal layers—the Backus average, even in its modified form, is not valid, due to its underlying assumption of a load on the top and bottom only.

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## Appendix A

layer	$c_{1111}$	$v_P$
1	10.56	3.25
2	20.52	4.53
3	31.14	5.58
4	14.82	3.85
5	32.15	5.67
6	16.00	4.00
7	16.40	4.05
8	18.06	4.25
9	31.47	5.61
10	17.31	4.16

Table 1: Density-scaled elasticity parameters, whose units are  $10^6 \text{ m}^{-2} \text{ s}^{-2}$ , for a stack of isotropic layers, and the corresponding  $P$ -wave speeds in  $\text{km s}^{-1}$ .

layer	$d_i$	$w_i$
1	115.47	0.0773
2	139.45	0.0934
3	195.07	0.1306
4	124.12	0.0831
5	204.61	0.1370
6	126.88	0.0849
7	127.85	0.0855
8	132.17	0.0885
9	198.04	0.1326
10	130.17	0.0871

Table 2: Distances,  $d_i$ , in meters, travelled by the  $P$  wave in each layer, and the corresponding averaging weights,  $w_i = d_i / (\sum_{j=1}^{10} d_j)$ .